Correlative amplitude–operational phase entanglement embodied by the EPR-pair eigenstate $|\eta\rangle$

2002 J. Phys. A: Math. Gen. 35 1007

(http://iopscience.iop.org/0305-4470/35/4/313)

J. Phys. A: Math. Gen. **35** (2002) 1007–1012 PII: S0305-4470(02)28238-4

Correlative amplitude–operational phase entanglement embodied by the EPR-pair eigenstate $|\eta\rangle$*

Hongyi Fan

CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China
and
Department of Material Science and Engineering, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China

Received 20 August 2001, in final form 20 November 2001
Published 18 January 2002
Online at stacks.iop.org/JPhysA/35/1007

Abstract
After a penetrating analysis of the known number–phase commutative relations we find that the common eigenvector $|\eta\rangle$ of two particles' relative coordinate and total momentum also embodies the entanglement with respect to the correlative amplitude–operational phase. Based on the fact that the number-difference operator is the canonical conjugate to the Noh–Fougères–Mandel operational phase operator, the corresponding entanglement is also briefly discussed.

PACS numbers: 03.65.Ud, 03.67.−a

In 1935, Einstein, Podolsky and Rosen (EPR) [1] published a paper arguing the incompleteness of quantum mechanics, in this paper they presented the conception of entanglement. They noticed the fact that two particles, although not interacting, are still entangled since their quantum state does not factor into a product of the states of each particle. Now the entangled states have been widely applied to quantum computation, quantum teleportation, quantum cryptography and quantum superdense coding [2–5]. In an entangled quantum state, measurement performed on one part of the system provides information on the remaining part. By observing the commutator $[X_1 - X_2, P_1 + P_2] = 0$, EPR introduced the wavefunction of a pair of particles with relative coordinate x_0:

$$\psi(x_1, x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p\, \mathrm{e}^{\mathrm{i}p(x_1 - x_2 + x_0)}$$

which describes a sharply correlated two-particle system. This $\psi(x_1, x_2)$, when projected on the momentum wavefunction $u_p(x_1) = \mathrm{e}^{\mathrm{i}px_1}/\sqrt{2\pi}$ of particle 1, yields $\psi_p(x_2) = \mathrm{e}^{\mathrm{i}p(-x_2+x_0)}/\sqrt{2\pi}$, because

$$\psi(x_1, x_2) = \int_{-\infty}^{\infty} \mathrm{d}p\, u_p(x_1)\psi_p(x_2).$$

* This work was supported by the President Foundation of the Chinese Academy of Science and the Educational Ministry of China.

On the other hand, projecting $\psi(x_1, x_2)$ on the coordinate eigenfunction $v_x(x_1) = \delta(x - x_1)$ of particle 1, yields collapse to $\psi_x(x_2) = \delta(x - x_2 + x_0)$, as

$$\psi(x_1, x_2) = \int_{-\infty}^{\infty} \mathrm{d}x\, \psi_x(x_2) v_x(x_1).$$

Thus there is a mysterious nonlocal entanglement between the separated quantum objects. The EPR argument has stimulated many discussions on the nonlocality and entanglememt inherent in quantum mechanics. In [6] we have constructed the common eigenvectors $|\eta\rangle$ of the two particles' relative position operators $X_1 - X_2$ and their total momentum $P_1 + P_2$ in two-mode Fock space, that is

$$|\eta\rangle = \exp\left[-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right]|00\rangle \tag{1}$$

where $\eta = \frac{1}{\sqrt{2}}(\eta_1 + \mathrm{i}\eta_2) = |\eta|\mathrm{e}^{\mathrm{i}\varphi}$ is a complex number, $|00\rangle$ is the two-mode vacuum state, $(a_i, a_i^\dagger)$, $i = 1, 2$, are the two-mode Bose annihilation and creation operators in the Fock space, $[a_i, a_j^\dagger] = \delta_{i,j}$. The $|\eta\rangle$ state satisfies the completeness relation

$$\int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\,\langle\eta| = 1 \qquad \mathrm{d}^2\eta \equiv \frac{1}{2}\mathrm{d}\eta_1\mathrm{d}\eta_2 \tag{2}$$

and possesses the orthonormal property

$$\langle\eta'|\eta\rangle = \pi\delta(\eta - \eta')\delta(\eta^* - \eta'^*).$$

$|\eta\rangle$ also obeys the eigenvector equations

$$(a_1 - a_2^\dagger)|\eta\rangle = \eta|\eta\rangle \qquad (a_2 - a_1^\dagger)|\eta\rangle = -\eta^*|\eta\rangle. \tag{3}$$

It then follows from $X_i = \frac{1}{\sqrt{2}}(a_i + a_i^\dagger)$, $P_i = \frac{1}{\mathrm{i}\sqrt{2}}(a_i - a_i^\dagger)$ that

$$(X_1 - X_2)|\eta\rangle = \eta_1|\eta\rangle \qquad (P_1 + P_2)|\eta\rangle = \eta_2|\eta\rangle. \tag{4}$$

From equation (4) we note a very important fact: although $[X_1, P_1] = [X_2, P_2] = \mathrm{i}$, the commutator $[X_1 - X_2, P_1 + P_2] = 0$ indicates the existence of coordinate–momentum entanglement between the two particles. According to the standard Schmidt decomposition theory (see [7]) for any pure state $|\Psi\rangle_{AB}$ of a bipartite system, there are orthonormal bases $\{|i\rangle_A\}$ for particle A and $\{|i'\rangle_B\}$ for particle B such that

$$|\Psi\rangle_{AB} = \sum_i \sqrt{p_i}|i\rangle_A|i'\rangle_B \tag{5}$$

if the number of nonvanishing eigenvalues (p_i') is greater than one, $|\Psi\rangle_{AB}$ is said to be entangled. Equation (5) is called the Schmidt decomposition of $|\Psi\rangle_{AB}$. The basic ingredient of the $|\eta\rangle$ state regarding the coordinate–momentum entanglement is demonstrated through its Schmidt decomposition process, i.e.

$$|\eta\rangle = \mathrm{e}^{-\mathrm{i}\eta_1\eta_2/2}\int_{-\infty}^{\infty}\mathrm{d}x|x\rangle_1 \otimes |x - \eta_1\rangle_2 \mathrm{e}^{\mathrm{i}x\eta_2} \tag{6}$$

where $|x\rangle_i$ is the coordinate eigenstate of X_i; or

$$|\eta\rangle = \mathrm{e}^{\mathrm{i}\eta_1\eta_2/2}\int_{-\infty}^{\infty}\mathrm{d}p|p\rangle_1 \otimes |-p + \eta_2\rangle_2 \mathrm{e}^{-\mathrm{i}p\eta_1} \tag{7}$$

where $|p\rangle_i$ is the momentum eigenstate of P_i. Equation (6) ((7)) implies that once particle 1 is measured in its coordinate eigenvector $|x\rangle_1$(momentum eigenvector $|p\rangle_1$), particle 2, no matter how far it is from particle 1, will simultaneously collapse into its coordinate eigenvector

$|x-\eta_1\rangle_2$ (momentum eigenvector $|-p+\eta_2\rangle_2$). This decomposition can also be realized by noting

$$\begin{aligned}\mathrm{e}^{\mathrm{i}P_1X_2}|\eta\rangle &= \mathrm{e}^{-\mathrm{i}\eta_1\eta_2/2}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}P_1(x_1-\eta_1)}|x\rangle_1\otimes|x-\eta_1\rangle_2\mathrm{e}^{\mathrm{i}x\eta_2}\\ &= \sqrt{2\pi}\mathrm{e}^{\mathrm{i}\eta_1\eta_2/2}|x=\eta_1\rangle_1\otimes|p=\eta_2\rangle_2\end{aligned}\tag{8}$$

thus

$$|\eta\rangle = \mathrm{e}^{-\mathrm{i}P_1X_2}\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\eta_1\eta_2/2}|x=\eta_1\rangle_1\otimes|p=\eta_2\rangle_2\tag{9}$$

and we have the need to name $\mathrm{e}^{-\mathrm{i}P_1X_2}$, which is a unitary operator, the entangling operator, and name $|\eta\rangle$ the EPR-pair eigenstate.

Another way of looking at the entanglement involved in the $|\eta\rangle$ state is as follows. By introducing the two-variable Hermite polynomial [8]

$$H_{m,n}(\eta,\eta^*) = \sum_{l=0}^{\min(m,n)}\frac{m!n!}{l!(m-l)!(n-l)!}(-1)^l\eta^{m-l}\eta^{*(n-l)}\tag{10}$$

and its generating function

$$\exp[-tt'+\eta t+\eta^*t'] = \sum_{m,n=0}^{\infty}\frac{t^mt'^n}{m!n!}H_{m,n}(\eta,\eta^*)\tag{11}$$

$|\eta\rangle$ can then be expanded in two-mode Fock space as (another form of the Schmidt decomposition)

$$|\eta\rangle = \sum_{m,n=0}^{\infty}\mathrm{e}^{-\frac{1}{2}|\eta|^2}\frac{1}{\sqrt{m!n!}}H_{m,n}(\eta,\eta^*)|m\rangle_s|n\rangle_i.\tag{12}$$

In this study, we explore if there exists any entanglement with respect to the phase and amplitude in the EPR-pair eigenstate $|\eta\rangle$.

Before addressing this question, let us see if one can use an ‘entangling’ operator $\exp\left(\mathrm{i}N_1\widehat{\mathrm{e}^{\mathrm{i}\theta_2}}\right)$, where $N_1 = a_1^\dagger a_1$ and $\widehat{\mathrm{e}^{\mathrm{i}\theta_2}}$ is the single-mode Susskind–Glogower (SG) phase operator [9]

$$\widehat{\mathrm{e}^{\mathrm{i}\theta_i}} = \frac{1}{\sqrt{N_i+1}}a_i = \sum_{n=0}^{\infty}|n\rangle_{ii}\langle n+1| \qquad i=1,2\tag{13}$$

to operate on $|\mathrm{e}^{\mathrm{i}\theta}\rangle_1\otimes|n\rangle_2$, as

$$\exp\left(\mathrm{i}N_1\widehat{\mathrm{e}^{\mathrm{i}\theta_2}}\right)|\mathrm{e}^{\mathrm{i}\theta}\rangle_1\otimes|n\rangle_2\tag{14}$$

to make up a number–phase entangled state? The answer is negative, because although the single-mode phase eigenstate is well defined [10]

$$|\mathrm{e}^{\mathrm{i}\theta}\rangle_1 = \exp\left[\mathrm{e}^{\mathrm{i}\theta}a_1^\dagger\sqrt{N_1+1}\right]|0\rangle_1 = \sum_{n=0}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\theta n}}{n!}\left(a_1^\dagger\sqrt{N_1+1}\right)^n|0\rangle_1 = \sum_{n=0}^{\infty}\mathrm{e}^{\mathrm{i}\theta n}|n\rangle_1\tag{15}$$

and

$$N_1|\mathrm{e}^{\mathrm{i}\theta}\rangle_1 = \sum_{n=0}^{\infty}n\mathrm{e}^{\mathrm{i}\theta n}|n\rangle_1 = -\mathrm{i}\frac{\partial}{\partial\theta}|\mathrm{e}^{\mathrm{i}\theta}\rangle_1\tag{16}$$

but $\widehat{\mathrm{e}^{\mathrm{i}\theta_2}}|n\rangle_2 = (1-\delta_{n,0})|n-1\rangle_2$ implies that $\widehat{\mathrm{e}^{\mathrm{i}\theta_2}}$ is not a unitary operator [10], so $\exp\left(\mathrm{i}N_1\widehat{\mathrm{e}^{\mathrm{i}\theta_2}}\right)$ is not unitary either, it cannot qualify as a number–phase entangling operator.

In a similar way to $[X_1 - X_2, P_1 + P_2] = 0$, one might consider

$$\left[\widehat{\mathrm{e}^{\mathrm{i}\theta_1}}\widehat{\mathrm{e}^{-\mathrm{i}\theta_2}}, N_1 + N_2\right] = 0 \tag{17}$$

as the starting point to construct a number–phase entangled state, here $\widehat{\mathrm{e}^{\mathrm{i}\theta_1}}\widehat{\mathrm{e}^{-\mathrm{i}\theta_2}}$ is the two-mode phase difference operator (each measurement of phase is actually measuring a phase difference between an objective light and a reference light) and exhibits its lowering and ascending behaviour on the state

$$|j, m\rangle = \frac{1}{\sqrt{(j+m)!(j-m)!}}|n_1 = j+m, n_2 = j-m\rangle \tag{18}$$

as

$$\widehat{\mathrm{e}^{\mathrm{i}\theta_1}}\widehat{\mathrm{e}^{-\mathrm{i}\theta_2}}|j, m\rangle = (1 - \delta_{m,-j})|j, m-1\rangle \qquad \widehat{\mathrm{e}^{-\mathrm{i}\theta_1}}\widehat{\mathrm{e}^{\mathrm{i}\theta_2}}|j, m\rangle = (1 - \delta_{m,j})|j, m+1\rangle. \tag{19}$$

However, $\widehat{\mathrm{e}^{\mathrm{i}\theta_1}}\widehat{\mathrm{e}^{-\mathrm{i}\theta_2}}$ is neither Hermitian nor unitary, we have to abandon considering the SG phase operator as our starting point for discussing the topic of number–phase entanglement.

Now we address if there exists any entanglement with respect to the phase and amplitude in the EPR-pair eigenstate $|\eta\rangle$. We still focus on the commutator $[X_1 - X_2, P_1 + P_2] = 0$, however, instead of considering the usual classical phase space composed of (x_i, p_i), we construct a phase space of $(x_1 - x_2)$ and $(p_1 + p_2)$, which means that we take $(x_1 - x_2)$ and $(p_1 + p_2)$ as the transversal and longitudinal axes, respectively, and $(x_1 - x_2) = 0$, $(p_1 + p_2) = 0$ as the origin in the space. Then the square of the radius $(x_1 - x_2)^2 + (p_1 + p_2)^2$ corresponds to the operator $(X_1 - X_2)^2 + (P_1 + P_2)^2$, and we can map the cosine of the rotating angle $\cos\varphi = (x_1 - x_2)/\sqrt{(x_1 - x_2)^2 + (p_1 + p_2)^2}$ in the space to an operator $\cos\Phi$

$$\cos\Phi = \frac{X_1 - X_2}{\sqrt{(X_1 - X_2)^2 + (P_1 + P_2)^2}}. \tag{20}$$

Using the relations $X_i = \frac{1}{\sqrt{2}}(a_i + a_i^\dagger)$, $P_i = \frac{1}{\mathrm{i}\sqrt{2}}(a_i - a_i^\dagger)$, $\cos\Phi$ can be explicitly expressed as

$$\cos\Phi = \frac{a_1 + a_1^\dagger - a_2 - a_2^\dagger}{2\sqrt{(a_1 - a_2^\dagger)(a_1^\dagger - a_2)}} = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\Phi} + \mathrm{e}^{-\mathrm{i}\Phi}) \tag{21}$$

where $\mathrm{e}^{\mathrm{i}\Phi}$ is

$$\mathrm{e}^{\mathrm{i}\Phi} \equiv \sqrt{\frac{a_1 - a_2^\dagger}{a_1^\dagger - a_2}}. \tag{22}$$

Note $[a_1 - a_2^\dagger, a_1^\dagger - a_2] = 0$, so they can reside in the same square root. Remarkably, since the common eigenvector of $a_1 - a_2^\dagger$ and $a_1^\dagger - a_2$ is just $|\eta\rangle$, $|\eta\rangle$ is also the eigenstate of $\mathrm{e}^{\mathrm{i}\Phi}$ with an eigenvalue being a phase

$$\mathrm{e}^{\mathrm{i}\Phi}|\eta\rangle = \left(\frac{\eta}{\eta^*}\right)^{\frac{1}{2}}|\eta\rangle = \mathrm{e}^{\mathrm{i}\varphi}|\eta\rangle \tag{23}$$

which tells us that in the $|\eta\rangle$ representation $\mathrm{e}^{\mathrm{i}\Phi}$ behaves as a phase operator. It is a happy coincidence that $\mathrm{e}^{\mathrm{i}\Phi}$ is just the operational phase operator proposed by Noh *et al* [11] and Freyberger *et al* [12] in their operational quantum phase measurement scheme with an eight-port homodyne detector. In [13] Hradil also suggested this phase operator. It is in [14, 15] that the explicit eigenstate $|\eta\rangle$ of $\mathrm{e}^{\mathrm{i}\Phi}$ was introduced. However, in [14, 15] the amplitude–phase entanglement involved in $|\eta\rangle$ was not touched upon. Using the completeness relation of $|\eta\rangle$ we have

$$\mathrm{e}^{\mathrm{i}\Phi} = \frac{\mathrm{d}^2\eta}{\pi}\mathrm{e}^{\mathrm{i}\varphi}|\eta\rangle\langle\eta|. \tag{24}$$

Similar to $a_1 = \sqrt{N_1+1}\mathrm{e}^{\mathrm{i}\theta_1}$ in (13), we can make the polar decomposition

$$a_1 - a_2^\dagger = \sqrt{\mathcal{A}^\dagger\mathcal{A}}\mathrm{e}^{\mathrm{i}\Phi} \qquad a_1^\dagger - a_2 = \mathrm{e}^{-\mathrm{i}\Phi}\sqrt{\mathcal{A}^\dagger\mathcal{A}}$$

we name $\mathcal{A}^\dagger\mathcal{A}$ the *correlative amplitude operator*

$$\mathcal{A}^\dagger\mathcal{A} = (a_1^\dagger - a_2)(a_1 - a_2^\dagger) = (X_1 - X_2)^2 + (P_1 + P_2)^2 \tag{25}$$

and have $[\mathrm{e}^{\mathrm{i}\Phi}, \mathcal{A}^\dagger\mathcal{A}] = 0$. From equation (3) we see

$$\mathcal{A}^\dagger\mathcal{A}|\eta\rangle = |\eta|^2|\eta\rangle \tag{26}$$

so $|\eta\rangle$ is also the common eigenvector of $\mathcal{A}^\dagger\mathcal{A}$ and $\mathrm{e}^{\mathrm{i}\Phi}$, in this sense we say that $|\eta\rangle$ also shows the entanglement in respect of the *correlation-amplitude operational phase*, a novel fact which has been unnoticed for a long time. Hence we can also name $|\eta\rangle$ the operational-phase state.

Since $\mathrm{e}^{\mathrm{i}\Phi}$ is unitary, from (21) we derive the angle operator

$$\Phi = \frac{1}{2\mathrm{i}}[\ln(a_1 - a_2^\dagger) - \ln(a_1^\dagger - a_2)] = \int \frac{\mathrm{d}^2\eta}{\pi}\varphi|\eta\rangle\,\langle\eta|. \tag{27}$$

In sharp contrast to the case of the usual (x_i, p_i) phase space, where the rectangular uncertainty relation $\Delta X_i \Delta P_i$ (a small square $\approx \hbar$) can be converted to the radius–angle uncertainty relation $R_i\Delta\theta_i\Delta R_i$ (a ring area in a small sector), which means that the radius R_i (corresponding to $\sqrt{x_i^2 + p_i^2}$) and the rotating angle θ_i cannot be precisely measured at the same time, we see that corresponding to the 'radius' and rotating 'angle' in the phase space of $(x_1 - x_2)$ and $(p_1 + p_2)$, the *correlative amplitude* operator commutes with Φ,

$$[\mathcal{A}^\dagger\mathcal{A}, \Phi] = 0 \tag{28}$$

which explains why $|\eta\rangle$ involves the *operational phase–correlative amplitude* entanglement.

To complete the theory of this type of entanglement of $|\eta\rangle$, we note that, similar to the SG operator $\left[N_1, \mathrm{e}^{\mathrm{i}\theta}\right] = -\mathrm{e}^{\mathrm{i}\theta}$, the two-mode number-difference operator $D = a_1^\dagger a_1 - a_2^\dagger a_2$ is the canonical conjugate to the angle operator Φ, $[D, \mathrm{e}^{\mathrm{i}\Phi}] = -\mathrm{e}^{\mathrm{i}\Phi}$. Moreover, while Φ corresponds to φ in the $|\eta\rangle$ representation, D behaves as

$$\begin{aligned} D|\eta\rangle &= [a_1^\dagger(\eta + a_2^\dagger) - a_2^\dagger(-\eta^* + a_1^\dagger)]|\eta\rangle = |\eta|\left(\mathrm{e}^{\mathrm{i}\varphi}a_1^\dagger + \mathrm{e}^{-\mathrm{i}\varphi}a_2^\dagger\right) \\ &\quad\times \exp\left[-\frac{1}{2}|\eta|^2 + |\eta|\left(\mathrm{e}^{\mathrm{i}\varphi}a_1^\dagger - \mathrm{e}^{-\mathrm{i}\varphi}a_2^\dagger\right) + a_1^\dagger a_2^\dagger\right]|00\rangle = -\mathrm{i}\frac{\partial}{\partial\varphi}|\eta\rangle \end{aligned} \tag{29}$$

so we must examine if any entanglement related to D exists. When we compare (29) with (16) and make reference to (15)

$$N_1|\mathrm{e}^{\mathrm{i}\theta}\rangle_1 = -\mathrm{i}\frac{\partial}{\partial\theta}|\mathrm{e}^{\mathrm{i}\theta}\rangle_1 \qquad |\mathrm{e}^{\mathrm{i}\theta}\rangle_1 = \sum_{n=0}^{\infty}\mathrm{e}^{\mathrm{i}\theta n}|n\rangle_1$$

we immediately realize that the operational-phase state $|\eta = |\eta|\mathrm{e}^{\mathrm{i}\varphi}\rangle$ can be expanded in its 'number-difference' basis:

$$|\eta = |\eta|\mathrm{e}^{\mathrm{i}\varphi}\rangle = \sum_{q=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\varphi q}|q, |\eta|\rangle \tag{30}$$

where the 'number-difference' basis $|q, |\eta|\rangle$ satisfies

$$D|q, |\eta|\rangle = q|q, |\eta|\rangle. \tag{31}$$

From (30) it is easily seen that

$$|q, |\eta|\rangle = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\,\mathrm{e}^{-\mathrm{i}\varphi q}|\eta = |\eta|\mathrm{e}^{\mathrm{i}\varphi}\rangle. \tag{32}$$

An entanglement involved in $|q, |\eta|\rangle$ may be analysed as follows. Note although

$$[D, (a_1 - a_2^\dagger)] = -(a_1 - a_2^\dagger) \qquad [D, a_1^\dagger - a_2] = a_1^\dagger - a_2 \tag{33}$$

we do have

$$[D, (a_1 - a_2^\dagger)(a_1^\dagger - a_2)] = [D, \mathcal{A}^\dagger \mathcal{A}] = 0 \tag{34}$$

so $|q, |\eta|\rangle$ is the common eigenvector of $\mathcal{A}^\dagger\mathcal{A}$ and D:

$$\mathcal{A}^\dagger \mathcal{A}|q, |\eta|\rangle = |\eta|^2 |q, |\eta|\rangle. \tag{35}$$

Together (31) and (35) show the entanglement between the *number-difference* and *correlative amplitude*. From (33) we see that $(a_1 - a_2^\dagger)$ and $(a_1^\dagger - a_2)$ are the lowering and ascending operators on the 'number-difference' basis:

$$(a_1 - a_2^\dagger)|q, |\eta|\rangle = |\eta||q-1, |\eta|\rangle \qquad (a_1^\dagger - a_2)|q, |\eta|\rangle = |\eta||q+1, |\eta|\rangle \tag{36}$$

and

$$\mathrm{e}^{\mathrm{i}\Phi}|q, |\eta|\rangle = |q-1, |\eta|\rangle \qquad \mathrm{e}^{-\mathrm{i}\Phi}|q, |\eta|\rangle = |q+1, |\eta|\rangle.$$

The above discussion can be extended to the state $|\xi\rangle$

$$|\xi\rangle = \exp\left[-\frac{1}{2}|\xi|^2 + \xi a^\dagger + \xi^* b^\dagger - a^\dagger b^\dagger\right]|00\rangle \tag{37}$$

which is the common eigenvector of $X_1 + X_2$ and $P_1 - P_2$,: one can similarly analyse the entanglement between

$$(a_1 + a_2^\dagger)(a_1^\dagger + a_2) \quad \text{and} \quad \sqrt{\frac{a_1 + a_2^\dagger}{a_1^\dagger + a_2}} \qquad D \quad \text{and} \quad (a_1 + a_2^\dagger)(a_1^\dagger + a_2)$$

without any difficulty.

In summary, we have plunged into the thick of the entanglement involved in the EPR-pair eigenstate. We have revealed that a new type of entanglement, correlative amplitude–operational phase entanglement, is inherent in the state $|\eta\rangle$, while $[D, \mathcal{A}^\dagger\mathcal{A}] = 0$ implies the *number-difference–correlative amplitude* entanglement. These are the new concepts of entanglement to which we should pay attention.

References

[1] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[2] Bennett C H *et al* 1993 *Phys. Rev. Lett.* **70** 1895
[3] Zeilinger A 1998 *Phys. World* **11** 35
[4] Ekert A and Josza R 1996 *Rev. Mod. Phys.* **68** 733
DiVinvenzo D P 1995 *Science* **270** 255
[5] Fuches C A *et al* 1997 *Phys. Rev.* A **56** 1163 and references therein
[6] Fan H-y and Klauder J R 1994 *Phys. Rev.* A **47** 704
Fan H-y and Fan Y 1996 *Phys. Rev.* A **54** 958
[7] Preskill J 1998 *Quantum Information and Computation (Lecture Notes for Physics vol 229)*
[8] Bateman Manuscript Project 1953 *Higher Transcendental Functions* ed A Erdelỳi (New York: McGraw-Hill)
[9] Susskind L and Glogower J 1964 *Physics* **1** 49
[10] Carruthers P and Nieto M M 1968 *Rev. Mod. Phys.* **40** 411
[11] Noh J W, Fougeres A and Mandel L 1992 *Phys. Rev. Lett.* **67** 1426
Noh J W, Fougeres A and Mandel L 1993 *Phys. Rev. Lett.* **71** 2578
Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press)
[12] Freyberger M, Heni M and Schleich W P 1995 *Quantum Semiclass. Opt.* **7** 187
[13] Fan H-y and Xiao M 1996 *Phys. Lett.* A **222** 299
[14] Fan H-y 2001 *J. Phys. A: Math. Gen.* **34** 1629
[15] Fan H-y and Sun Z 2000 *Phys. Lett.* A **272** 219